The speeding up of the arrowhead decomposition method*

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We present further development of the arrowhead decomposition method (ADM) for a fast parallel solution of the block-tridiagonal system of linear equations [1, 2]. The method is based on the ideas of domain decomposition [3,4] and consists in rearranging the original linear system into an equivalent one with the "arrowhead" structure of the matrix. Such a structure provides a good opportunity for parallel solving: it includes simultaneous solutions of the systems with large on-diagonal blocks and fast solving of the equation for the coupling matrix [5,6]. Initially, ADM employed the matrix sweeping (Thomas) algorithm [7] for dealing with the blocks of the arrowhead structure. It allowed us to observe an essential growth of a performance in comparison with the sequential matrix algorithms [2]. Moreover, we managed to analytically estimate the optimal number of parallel processors required to reach the maximum computational speedup. The method was used for a fast numerical solution of the boundary value problem for the two-dimensional integro-differential Faddeev equations to study neutron-deuteron scattering [8].

The current contribution is focused on the recursive application of ADM as well as employing the parallel (block) cyclic reduction (PCR) [9] to speed up the overall algorithm. A recursive application constitutes using ADM for solving the independent systems with large on-diagonal blocks. In this respect, an important issue is the allocation of parallel processors for each level of recursivity. We analytically estimate a computational speedup with respect to the sequential matrix sweeping algorithm based on the number of elementary multiplicative operations for the parallel and serial parts of the methods. We show that the recursive application gives a significant improvement of the performance of the entire algorithm over the initial ADM. However, to achieve a maximum speedup for a given linear system a number of parallel processors should be in several times larger.

Another important issue is a fast parallel solving of the equation for the coupling matrix. This equation is considered as a bottleneck of the decomposition methods [4] since it collects results of some parallel parts of the method and provides the data for further parallel solving. Thus, the way of solution of this equation significantly affects the performance. In our case the coupling matrix is block-tridiagonal, so we use PCR for its fast parallel solving. As a result, we show a notable improvement of the performance of the ADM with PCR over the original one. In particular, let us consider the original linear system with N blocks on the diagonal. If the number of parallel processors P equals to a number of large on-diagonal blocks of ADM, the compuational speedup S with respect to the sequential matrix sweeping algorithm is, in the leading order, given by

$$S = \frac{3P}{7} \frac{1}{\left(1 + \frac{P}{N} \log_2(P-1)\right)},$$

where $P-1$ is a number of blocks on the diagonal of the coupling matrix. The speedup almost linearly depends on a number of parallel processors, that clearly indicates better scalability of the ADM with PCR. We also compare our analytical estimations of the computational speedup with the practically obtained results.

In addition, we measured the computation time of ADM as well as of Intel MKL PARDISO and MUMPS 4.10 for the same block-tridiagonal linear systems. A significantly improved performance of our method in comparison with the mentioned state-of-art packages is observed.

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