Russian Supercomputing Days 2018

Extremely High-order Optimized Multioperators-based Schemes and Their Applications to Flow Instabilities and Sound Radiation

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DNS of multiscale transient problems: how to get accurate solutions with realistic numbers of grid points (generally, DOF)?

For required accuracy:

(1) Use efficient parallel codes low-order methods

(2) Decrease DOF by using high-order methods

3D unsteady CFD: n times DOF decrease for x,y,z $\longrightarrow$ n^4 decrease in operation count

(3) Use parallel codes for high-order methods

Present talk: use very high-order with (3)

General idea of constructing arbitrary-order accurate multioperators formulae (Tolstykh, 1997, Parallel CFD, Manchester)

Some papers: Tolstykh, JCP(2007,2008), Commun.in Comp.Phys.(2017)

А.И.Толстых Компактные и мультиоператорные аппроксимации высокой точности для уравнений в частных производных, М. Наука,2015

- **Consider compact approximation** $L_m(c)\,f_j = Lf\Big|_j + O(h^m)$

Let $c_1, c_2, \mathrm{K}\ c_M$ **be fixed values defining basis operators**

$$L_m(c_1), L_m(c_2), \mathrm{K}\ L_m(c_M)$$

Multioperator is $$L_M(c_1, c_2, \mathrm{K}\ c_M) = \sum_{k=1}^{M} \gamma_k L_m(c_k), \quad \sum_{i=1}^{M} \gamma_i = 1$$

where γ_k **satisfy a linear system which equations kill ($M-1$) low-order terms of the truncation error. The solvability can be proved or verified. It gives**

$$Lf\Big|_j = L_M f_j + O(h^{M+m-1})$$

Existence and example of construction

- The solvability of the linear system and hence existence and uniqueness can be proved if basis operators are compact approximations to target one

- Consider approximate formula $$[Lu]_j \approx L_h[u]_j = \sum_k c_k u_{j+k}$$

- Introduce operator $$A_h = I + cB_h, \quad B_h = O(h) \text{ или } B_h = O(h^2)$$

Form superposition $$\overline{L_h}(c) = A_h^{-1}(c) L_h$$

Fix $c_1, c_2, \ldots c_M$, solve for $\gamma_1, \gamma_2, \ldots \gamma_M$, obtain $$L_M(c_1, c_2, \ldots c_M) = \sum_{i=1}^{M} \gamma_i \overline{L_h}(c_i)$$

Parameters $c_1, c_2, \ldots c_M$ can be used to control the properties

Multioperators for fluid dynamics.

Recent version of basis operators with two-point inversions.

Very high orders, presently up to 36! (Tolstykh, Commun. In Comp. Phys.,2017)

Approximate $Lf_j = \left(\dfrac{\partial f}{\partial x}\right)_j$ by left and right operators

$L_l f_j = R_l(c)^{-1}(f_j - f_{j-1})\,,\ L_l > 0$ and $L_r f_j = R_r(c)^{-1}(f_{j+1} - f_j),\ L_r < 0$

where $R_l u_j = (1-c)u_j - cu_{j-1}$ and $R_r u_j = (1-c)u_j - cu_{j+1}$

Skew-symmetric, approximate derivatives

$$L_0(c) = \frac{1}{2}\big(L_l(c) + L_r(c)\big) = L + O(h^2) \Longrightarrow L_M(c_1,\ldots,c_M) = \sum_{k=1}^{M} \gamma_k L_0(c_k) f_j = (Lf)_j + O(h^{2M})$$

Self-adjoint positive, for dissipation

$$D(c) = \frac{1}{2}\big(L_l(c) - L_r(c)\big) = O(h) \longrightarrow D_M(c_1,\ldots,c_M) = \sum_{k=1}^{M} \gamma_k D(c_k) = O(h^{2M-1})$$

Phase & Amplitude Errors of 16th & 32nd - order schemes with two-diagonal inversions

$$u_t + au_x = 0, \quad u(0,x) = e^{ikx}$$

$$u(t,x) = e^{ik(x-at)}$$

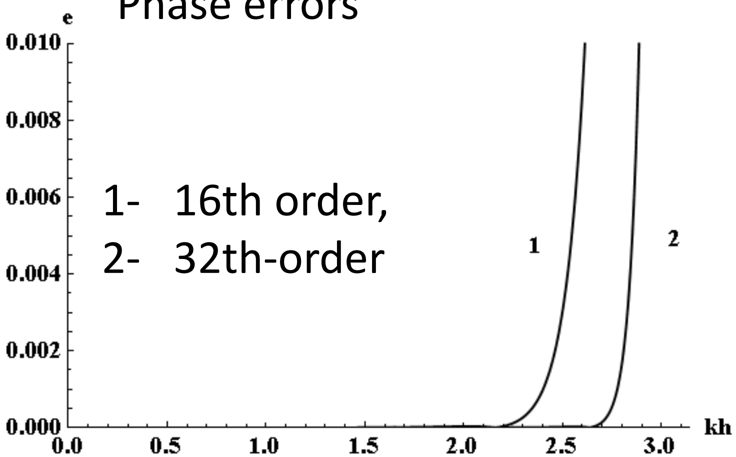
$$u_t + aL_M u/h = 0$$

$$u(t,x_j) = e^{-dt}e^{ik(x-a^*t)}$$

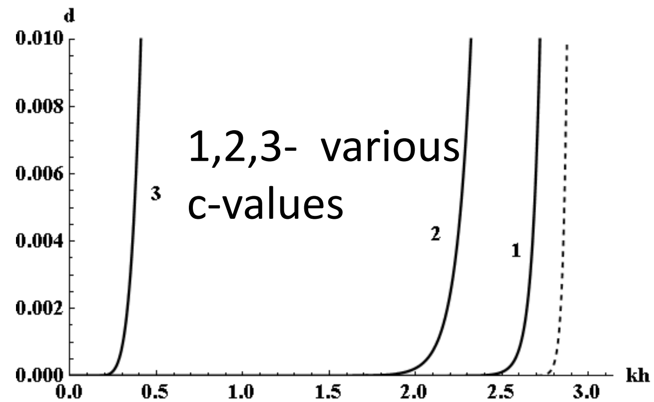
Phase error $$e(kh) = \left(a - a^*(kh; c_1, \ldots c_M)\right)/a$$

Controllable dissipation $d(kh)$

Phase errors



16th order



Architecture of multioperarors-based schemes

$$\partial u/\partial t + \partial f(u)/\partial x = 0$$

- Multioperators: specify $\boxed{M}$. Use preliminary analysis to specify $\boxed{c_1, c_2, \ldots c_M}$ and $\boxed{c'_1, c'_2, \ldots c'_M}$, create multioperators
- $L_M(c_1, c_2, \ldots c_M)$ and $D_M(c'_1, c'_2, \ldots c'_M)$
- Conservative scheme (can be put in the form of flux balances) $c'_i = c_i$

$$\partial u/\partial t + L_M f(u) + aD_M u = 0, \quad a = const \geq 0$$

Dissipation-free approximation to the derivative

High-order dissipation

Multidimensional problems: use multioperators for each spatial coordinate

N-S equations: use any type operators for viscous terms

Example: smooth solution of the Hopf equation

$$u_t + (u^2/2)_x = 0, \quad u(0,x) = 0.5 + \sin \pi x, \quad -1 \le x \le 1$$

10th (M=4) & 16^{th}(M=8) order schemes with two-diagonal inversions, C-norm of the solutions errors

	WENO-5		10^{th} order		16^{th} order	
N	error	order	error	order	error	order
16	1.3e-2		1.3e-3		1.3e-3	
32	1.2e-3	3.4	6.6e-6	7.7	8.5e-6	7.3
64	9.5e-5	3.7	5.4e-9	10.3	1.3e-9	12.6
128	3.3e-6	4.8	4.9e-12	10.1	3.7e-14	15.1
256	8.7e-8	5.3	8.1e-14	5.9		

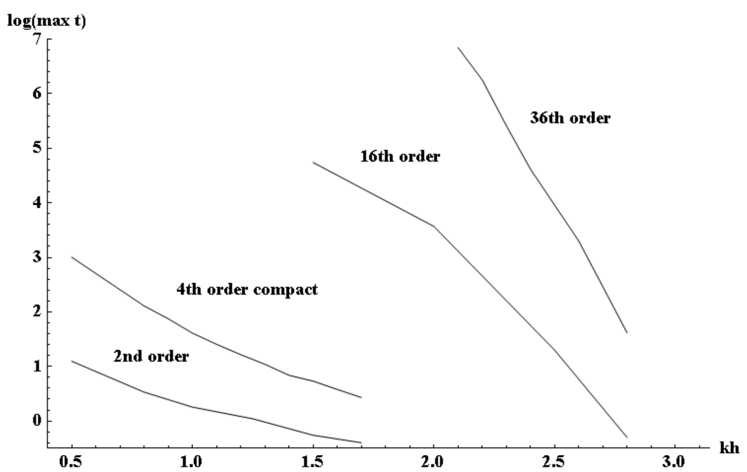
Benchmark problem (C. Tam),
32nd- order scheme with near-optimal values of c_i parameters

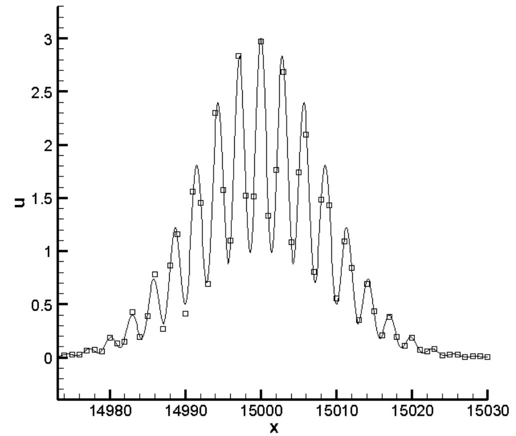
$$u_t + u_x = 0, \quad u(0,x) = [2 + \cos \beta x]\exp[(-\ln 2)(x/10)^2]$$

$h = 1$, required $\beta = 1.7, \quad t = 800$

c $\quad \beta = 2.2 \quad t = 15000$

Time needed to preserve 10% accuracy vs. wave number





32th order

Example of CFD parallel implementation. Left sweeps in x—direction $u_j = a(c_i)u_{j-1} + b(c_i), j = 1,2,\dots,N$

Partition $[O, x_N]$ into $[x_{m-1}, x_m]$, $m = 1,2,K,N$

Calc.Fk: performing left k – th sweep

Processor m: performing sweep for part $x_{m-1} \le x \le x_m$

$O \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \Lambda$

Proc.1	Proc.2	Proc.3	Proc.4	Λ
Calc.F1				
Calc.F2	Calc.F1			
	Calc.F2	Calc.F1		Λ
N		Calc.F2	Calc.F1	Λ
Calc.F(m-1)	N		Calc.F2	Λ
Calc.Fm	Calc.F(m-1)	Calc.F(m-2)	Λ	Λ
Calc.F(m+1)	Calc.Fm	Calc.F(m-1)	Calc.F(m-2)	Calc.F(m-3)

Execution times per time step in 3D case (jets) mesh 360x100x100 Lomonosov sup.comp.

Number of proc.	8	27	64	125	216	360	1000
Distrib. Along axes	2x2x2	3x3x3	4x4x4	5x5x5	6x6x6	6x10x6	10x10x10
Time per time step,sec.	113	27.45	6.34	3.99	2.81	2.81	1.70
Accelerati on	1	4.12	8.94	17.8	28.3	40.3	66.5

Target problems

- **(I). Steady state problems (smooth meshes are required)**

We are interested in:

- **(II) Unsteady problems requiring long-time integrations with preserving high resolution of small scales**

- Aeroacoustics DNS (instability with sound radiation)
- DNS of turbulence, laminar-turbulent transition
- 3D unstable vortex wakes generated by landing large aircrafts
- Atmospheric phenomena (e.g., tornado)
- Many others

Using high-order multioperators –based schemes, it is possible to catch fine details of flows using the Navier-Stokes equations with modest meshes

Direct simulation of unstable subsonic hot axisymmetric jets: getting fine details

- Unsteady Navier-Stokes equations
- 10^{th}-16^{th} order multioperators schemes

detect

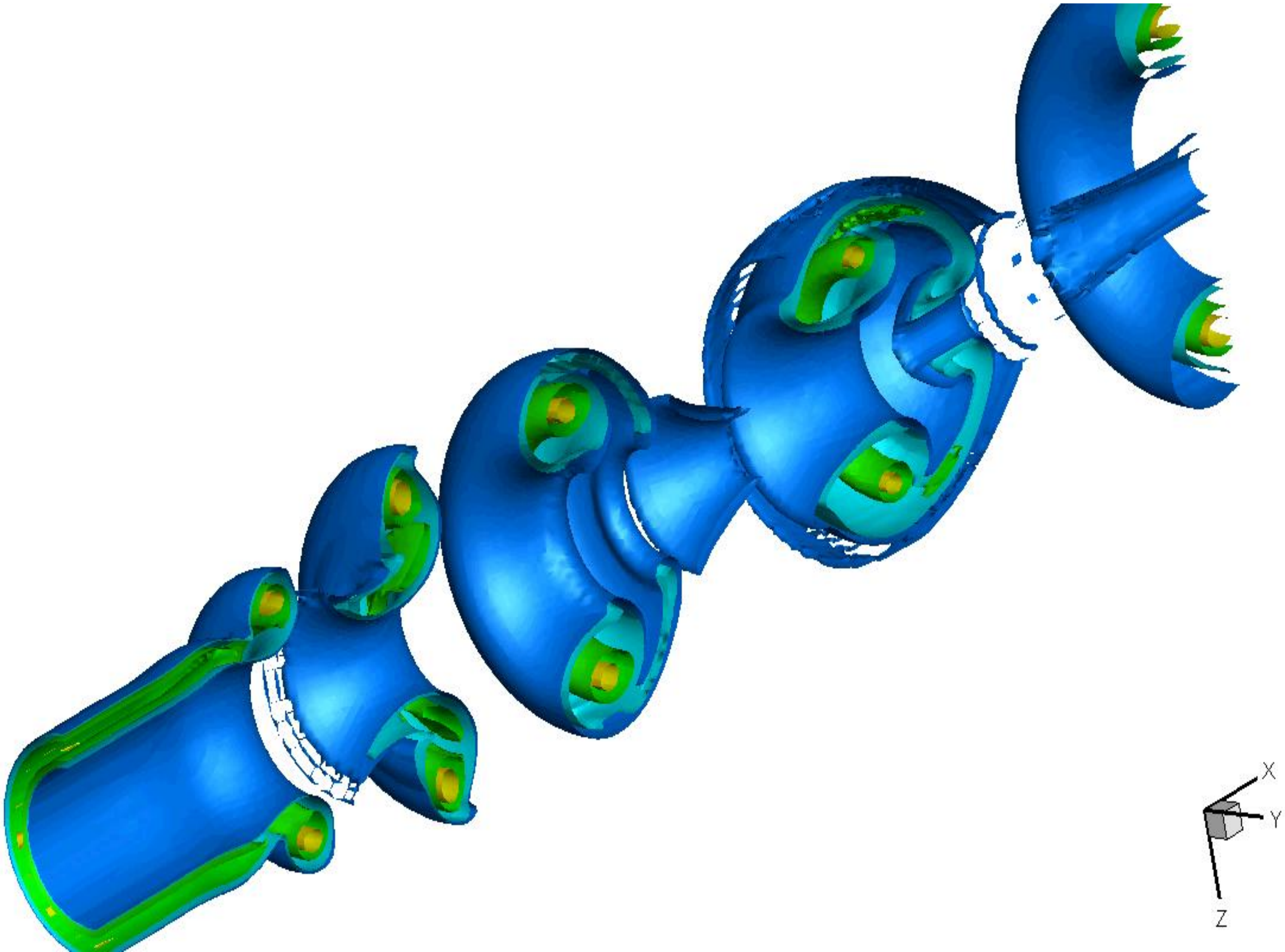
Fine details of vortex rings formation, their interactions and break down,

describe

Sound radiation and its origin

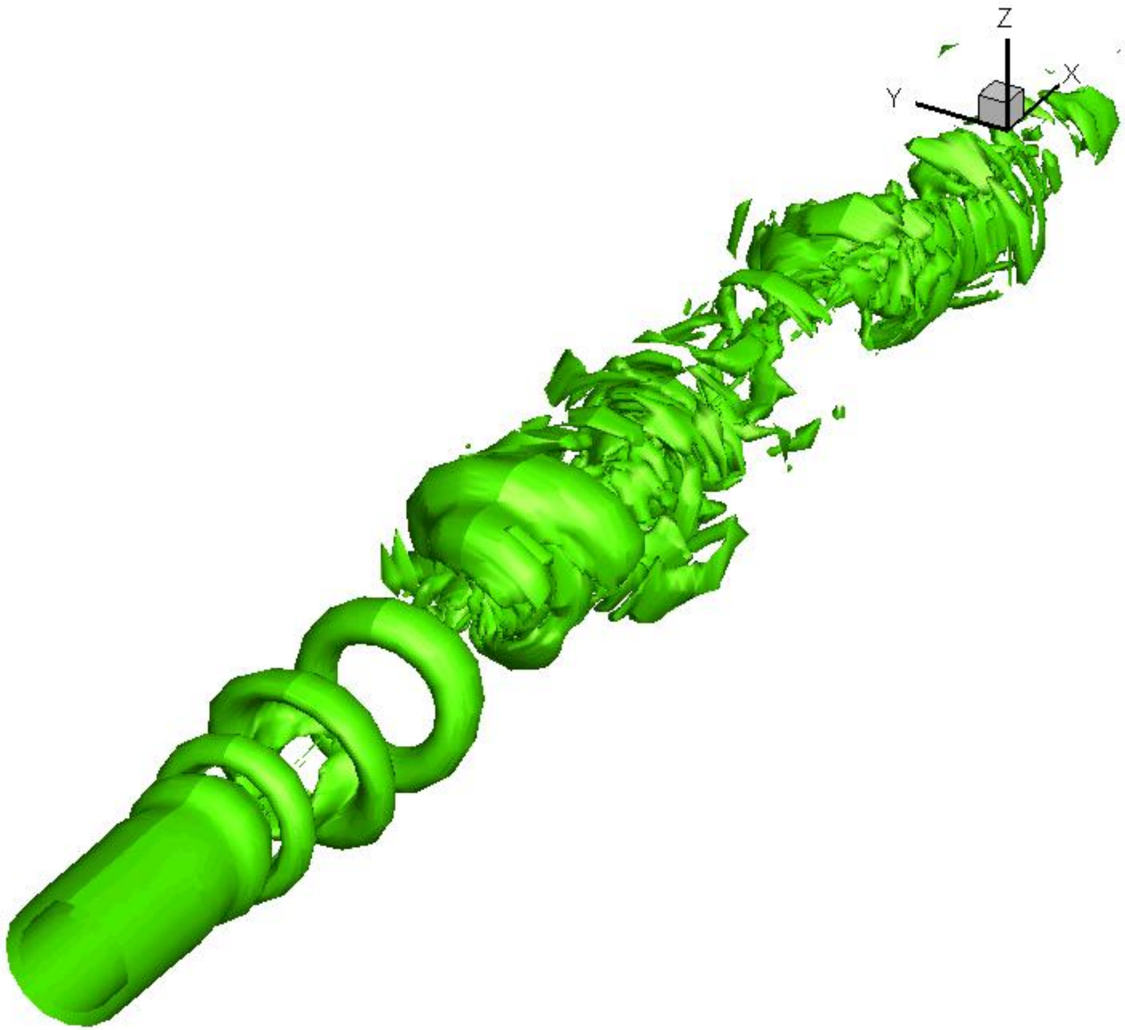
Cold jet. Axisymmetric formulation. M=0.5

3d view of the vorticity field (fragment)

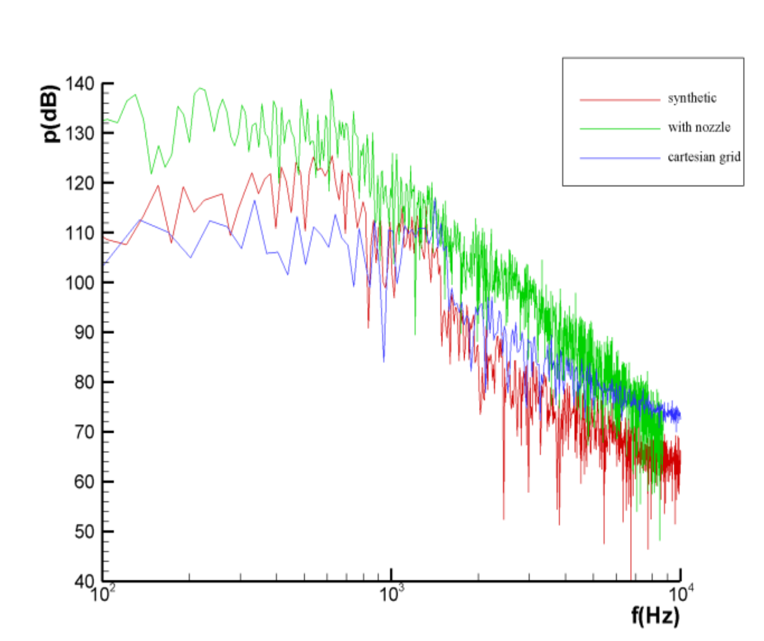


3D hot jet , M=0.1 $T/T_\infty = 2.5$ Abs. values of vorticity, 4<x<30.

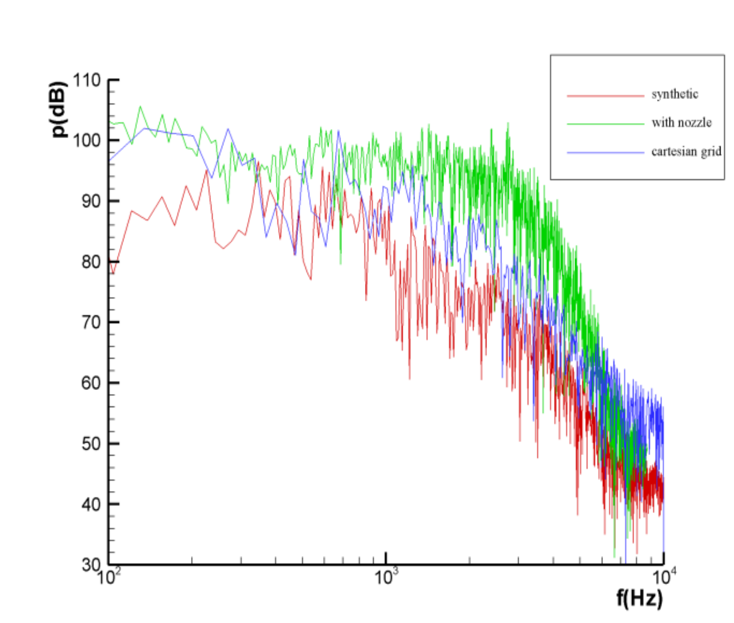
Azimuthally modes
spoil vortex rings



Spectra examples



$d = 20R, \quad \alpha = 10^{\circ}$



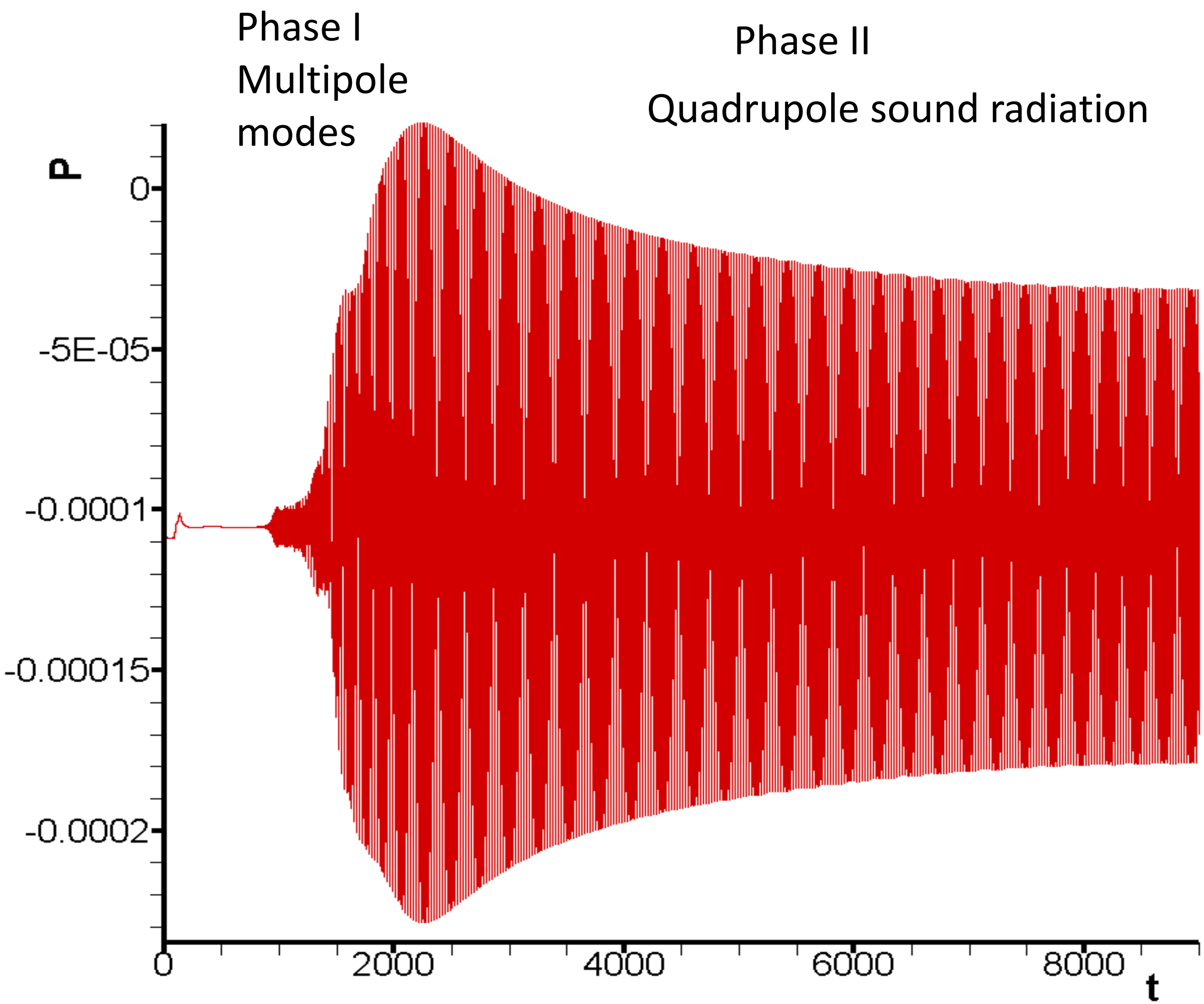
$d = 20R, \quad \alpha = 40^{\circ}$

Instability of Rankine vortex in compressible gas with sound radiation

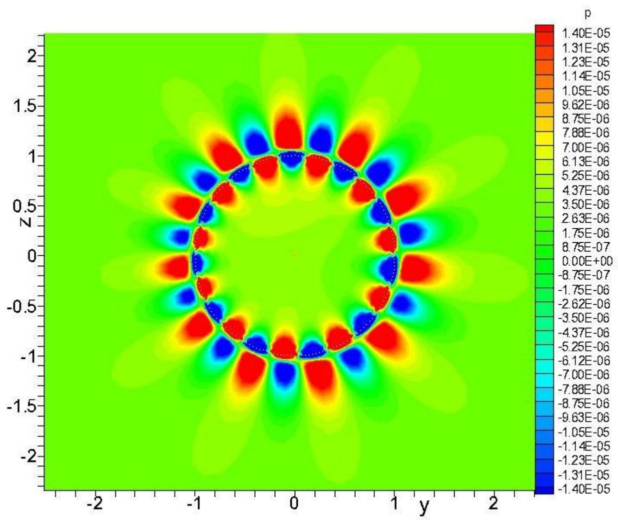
- Vorticity : $$\Omega = const, \quad r \le R, \qquad \Omega = 0, \quad r > R$$
  $$M = \frac{\Omega R}{2c_\infty}$$
- Incompressible case: the velocity field is exact solution which is stable in respect to small perturbations
- Compressible case: the velocity field is exact solution but it is unstable

- The problem: numerical simulation of the instability scenario using the Euler equations

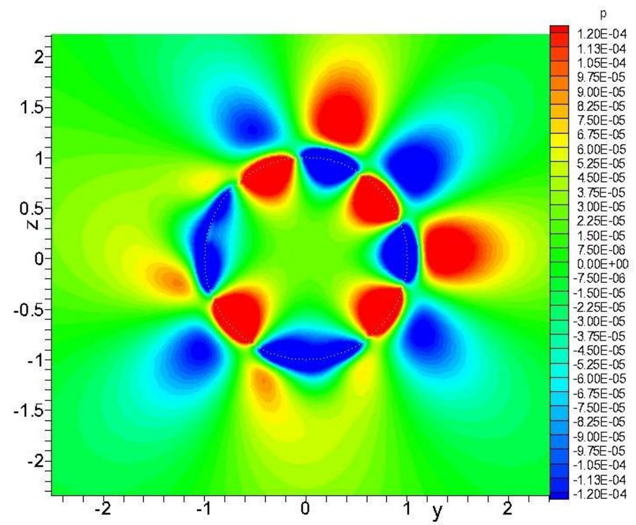
Time history of pressure pulsations at R=20, M=0.3



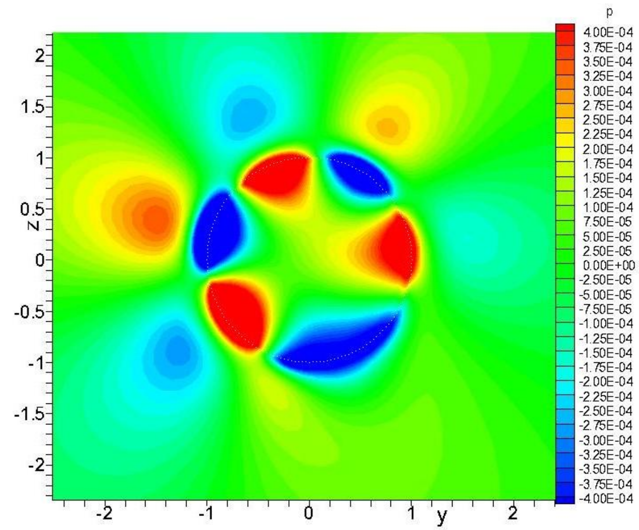
Snapshots of acoustic pressure fields near the vortex boundary. Phase I.



T=700

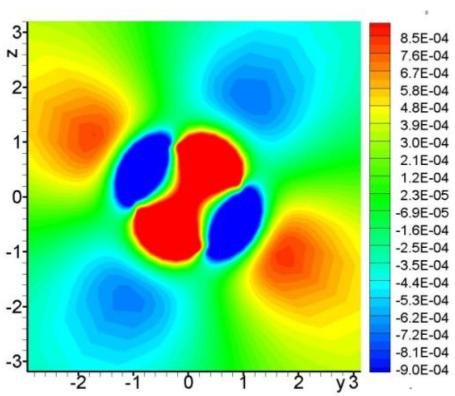


T=1380

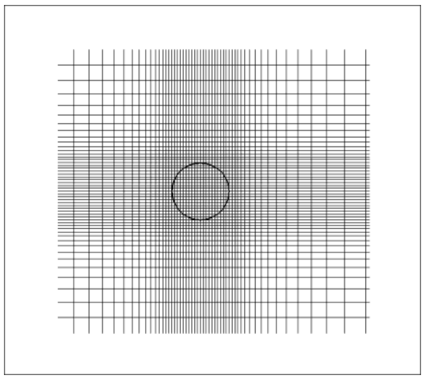


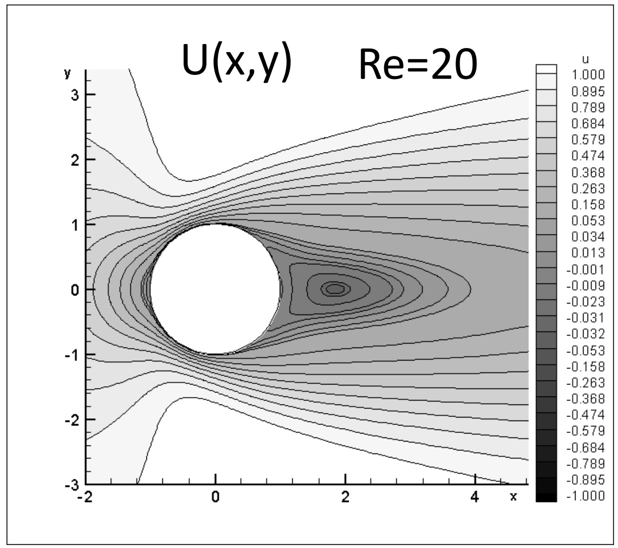
T=2500

Phase II

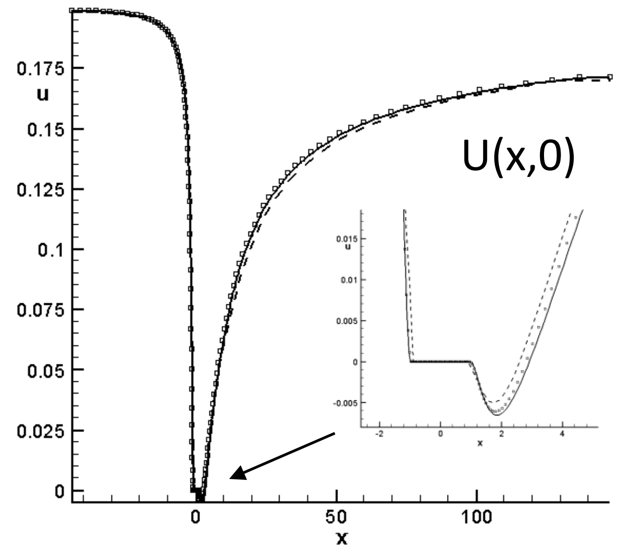


Using Immersed Boundaries Method. Test: flow about cylinder, small Re, M=0.2





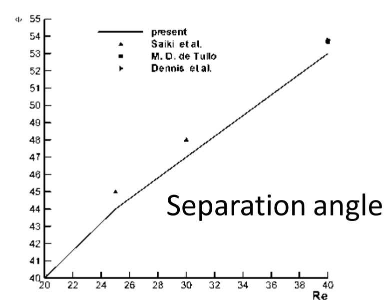
Dashed: 88x80, markers: 352x320

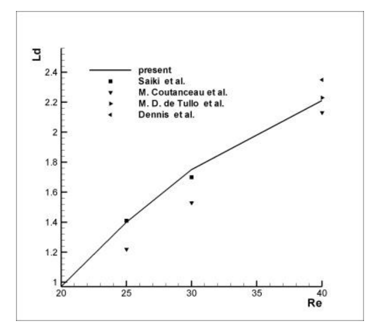


Tested options: u=v=0 in cyl., forcing outside and inside cyl.

↓

No visible influence on near and far fields

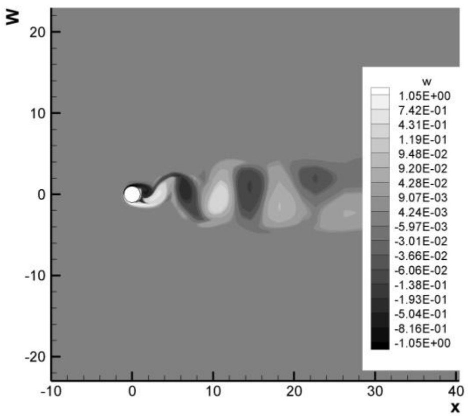




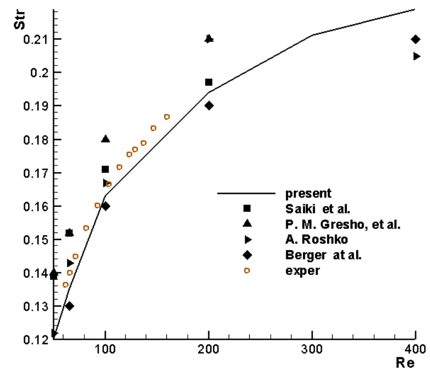
Length of separ. zone

IBM. Test: flow about cylinder, Re>40, M=0.2

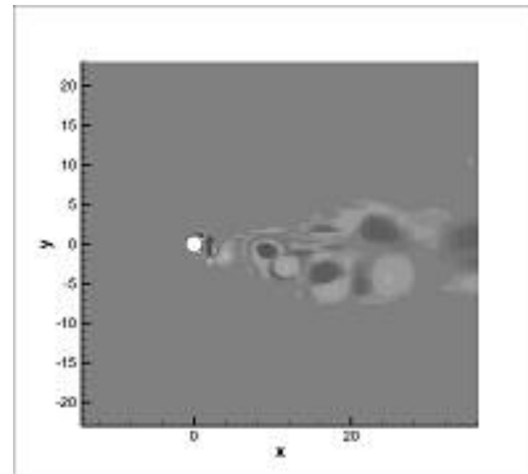
Vorticity, Re=400



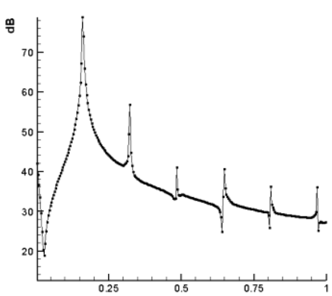
Strouhal numbers



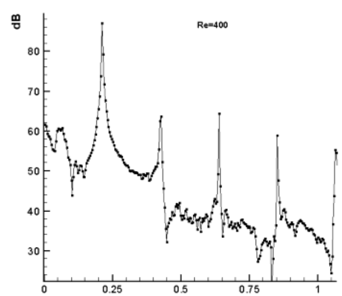
Vorticity, $\mathrm{Re} = 10^8$



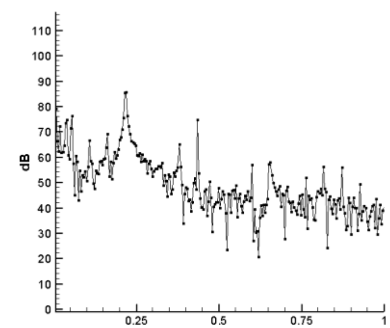
Spectra at x=-4.9, y=15.2



Re=100



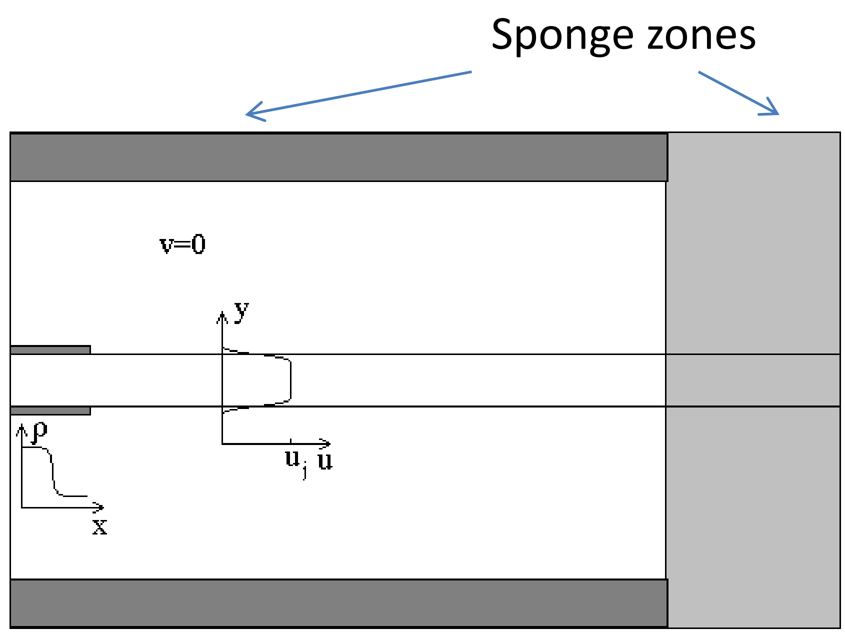
Re=400



$\mathrm{Re} = 10^8$

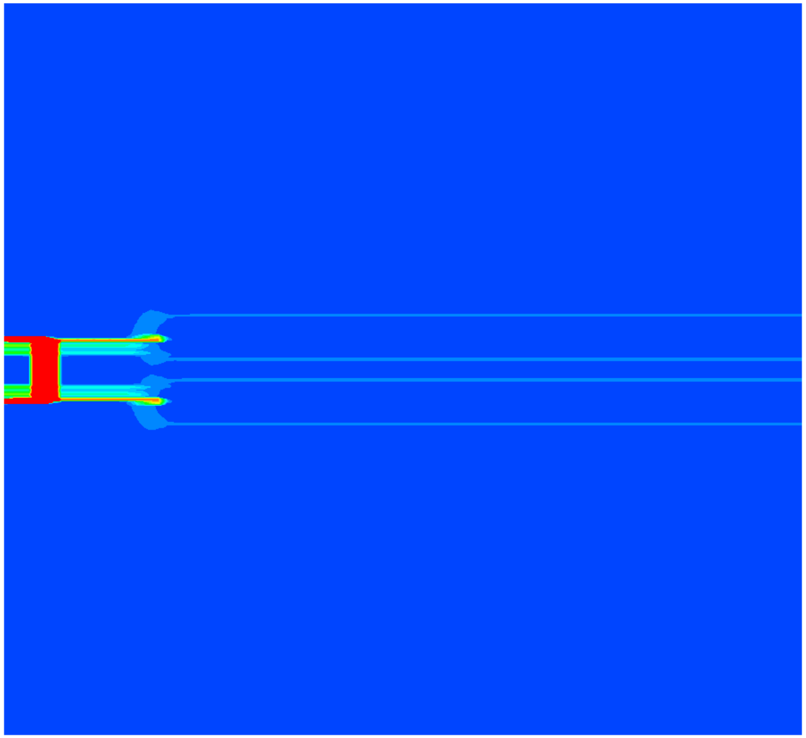
Calculations for supersonic flows (M<1.5) are possible due to conservative property of multioperators-based schemes

- The schemes can deal with shock-capturing calculations . Example: underexpanded supersonic jets. Screech effect (upwind propagations of acoustic waves}.



Underexpanded jet, M=1.5. Screech effect.

Schlieren visualization (abs of density gradients) of the flow field



Flows with strong shocks and contacts. Hybrid multioperators schemes.

- Main idea: to get monotone solutions near shocks and contacts regions and high-order ones away from those regions.
- Tools:

-- Using flux corrections (Zalesac, J.Comp.Phys,1979) and/or

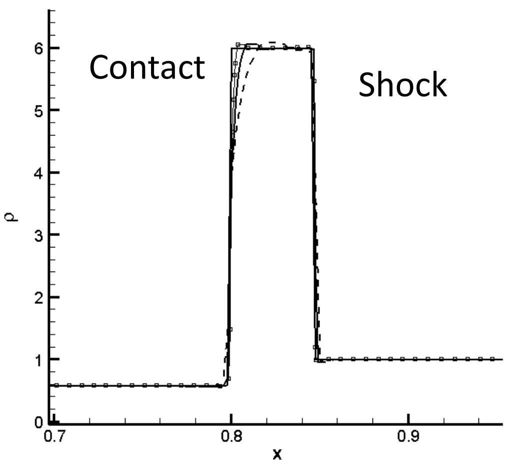
-- Blending high-order and monotone schemes

(I.B.Petrov, A.S.Kholodov, Comput. Math. Math. Phys.,1984;

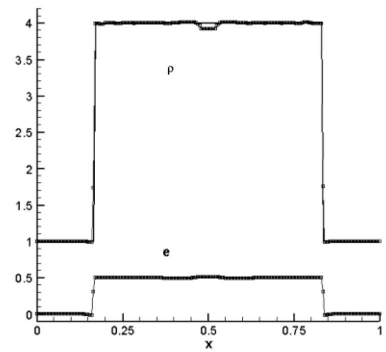
M. N. Mikhailovskaya , B. V. Rogov, Comput. Math. Math. Phys., 2012)

High Mach numbers, 16th-order hybrid

Riemann problems

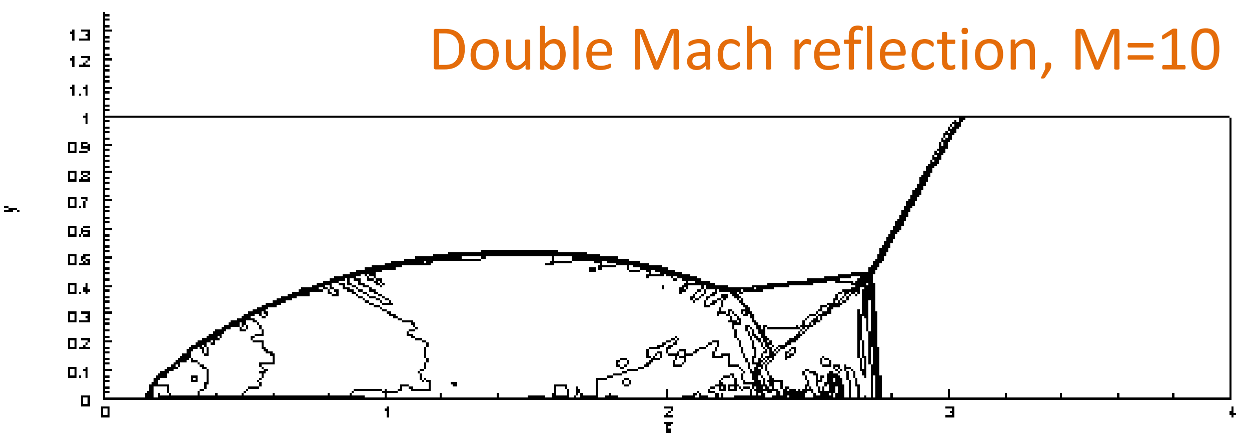


Toro problem,
M=198



Noh problem,
flow collision,
M~1000

Double Mach reflection, M=10



Conclusions

- Using the multioperators approach, it is possible to create desired-order approximations for numerical analysis formulae
- 10 th –32th - order multioperators-based optimized schemes for fluid dynamics were constructed
- Extremely high accuracy and high resolution was demonstrated using benchmark problems
- The potential for efficient massively parallel calculations does exist
- High fidelity direct NS and Euler calculations of sound generation due to flow instabilities were carried out
- The schemes can deal with shock-capturing calculations
- Hybrid schemes can be used in the case of strong shocks and hypersonic flows

Thank you