Numerical Method for Solving a Diffraction Problem of Electromagnetic Wave on a System of Bodies and Screens

Mikhail Medvedik[0000−0003−4066−1818], Marina Moskaleva[0000−0002−4501−8744]✉ and Yury Smirnov[0000−0001−9040−628X]

Penza State University, Penza, Russia
_medv@mail.ru, m.a.moskaleva1@gmail.com, mmm@pnzgu.ru

Abstract. The three-dimensional vector problem of electromagnetic wave diffraction by systems of intersecting dielectric bodies and infinitely thin perfectly conducting screens of irregular shapes is considered. The original boundary value problem for Maxwell‘s equations is reduced to a system of integro-differential equations. Methods of surface and volume integral equations are used. The system of linear algebraic equations is obtained using the Galerkin method with compactly supported basis functions. The subhierarchical method is applied to solve the diffraction problem by scatterers of irregular shapes. Several results are presented. Also we used a parallel algorithm.

Keywords: Boundary value problem · Inverse problem of diffraction · Permittivity tensor · Tensor Green’s function · Integro-differential equation.

Introduction

The important area in modern electrodynamics is three-dimensional problems of electromagnetic wave diffraction on systems of dielectric bodies and infinitely thin perfectly conducting screens of various shapes. In such diffraction problems, it is necessary to find solutions of Maxwell‘s equations that satisfy certain boundary or transmission conditions and radiation conditions at infinity.

Works of A. A. Samokhin [1] and M. Costabel [2,3] are devoted to problems of diffraction of electromagnetic waves by dielectric bodies. In these works, as well as in the work of D. Colton and R. Kress [4], a theory of solvability of vector diffraction problems is developed: existence and uniqueness of the solutions are proved and numerical methods are presented.

A theory of solvability of three-dimensional electrodynamics problems on nonclosed surfaces is developed by A. S. Ilyinsky and Yu. G. Smirnov [5]. Existence and uniqueness of the solution (in suitable spaces) are proved.

In papers [6–8] the diffraction problem of electromagnetic wave by screens of various forms is considered. The statement of the boundary-value problem of the diffraction of an electromagnetic wave by a system of nonintersecting bodies and screens is presented in [9, 10].

It should be noted that the theory of solvability of electromagnetic diffraction problems on a system of dielectric bodies and infinitely thin perfectly conducting screens is far from completion. However, recent advantages in this theory have been presented in [9, 10] for the case of nonintersecting bodies and screens.

In this paper we present numerical results of the solution of the diffraction problem on a system of intersecting bodies and screens, which is new in comparison with [5–11].

Let us consider various numerical methods for solving electromagnetic waves diffraction problems on bodies of various configurations, such as finite element methods (FEM), finite-difference methods (FDM) and methods of surface and volume integral equations. FEM and FDM also are used in various application package for solving electrodynamic problems. However, these methods have some disadvantages. For example, its application is possible only if the region in which the problem is solved is made finite. This limitation of the region leads to incorrect results. In order to avoid this, it is necessary to artificially increase the size of the region in which the problem is solved. Such algorithm leads to the appearance of sparse matrices of sufficiently large order $(10^5 - 10^6)$. The boundary value problem also is not elliptic in the general case. Therefore, the use of traditional method of proof of convergence of projection methods is excluded. Alternatively, the method of surface and volume integro-differential equations, free from described above disadvantages, can be applied. Then the equation is solved in the region of inhomogeneity inside the body and on the screen. Thus, after the discretization of the problem, we obtain a finite-dimensional system of equations with a dense matrix of order $(10^3 - 10^4)$, that is substantially smaller than in the case FEM or FDM.

In this paper we apply methods of surface and volume integro-differential equations for the numerical solution of the diffraction problem on a system of intersecting bodies and screens. Also we use the subhierarchical approach [6, 7] to solve the problem on a system of intersecting bodies and screens of complex shapes. The problem is reduced to a system of integro-differential equations. The system of linear algebraic equations is obtained using the Galerkin method. Solutions of the diffraction problem of electromagnetic waves on a system consisting of a screen that is intersected with an inhomogeneous body are presented.

1 Statement of the Problem

Let Θ be a system of infinitely thin perfectly conducting screens Ω_k $k = 1 \dots K$ and dielectric bodies Q_j $j = 1 \dots J$. The system Θ is placed in $\mathbb{R}^3$. An example of Θ, where $K = 1$, $J = 1$, is shown in figure 1.

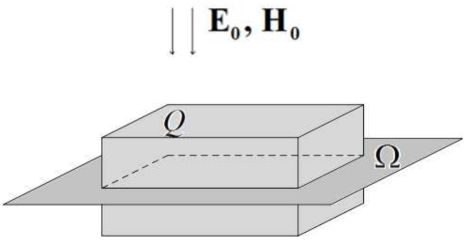

Fig. 1. The body Q and the screen Ω.

The screens Ω_k are connected orientable nonclosed disjoint bounded surfaces of class C^∞ in $\mathbb{R}^3$ [5]. The boundary $\partial\Omega_k := \overline{\Omega}_k \setminus \Omega_k$ of screen Ω_k is a piecewise smooth curve that consist of a finite number of simple arcs without self-intersections. We use the notation $\Omega = \cup_k \Omega_k$, and $\partial\Omega = \cup_k \partial\Omega_k$.

We assume that Q_j is a bounded domain having the boundary $\partial Q_j := \overline{Q}_j \backslash Q_j$, $j = 1 \dots J$. It is assumed the ∂Q_j is a piecewise smooth closed orientable surface consisting of a finite number of surfaces of class C_1. The bodies Q_j can be inhomogeneous and anisotropic. Thus the inhomogeneity is described by the tensor

$$\hat{\varepsilon} = \begin{cases} \varepsilon_e \hat{I}, & x \in \mathbb{R}^3 \setminus (\bar{Q} \bigcup \bar{\Omega}), \\ \hat{\varepsilon}_j(x), & x \in \bar{Q}_j, \end{cases}$$

$\hat{\varepsilon}_j = \hat{\varepsilon}_j^T$, $\operatorname{Im} \hat{\varepsilon}_j \geq 0$.

The screens Ω_k and bodies Q_j are intersecting.

Let $\mathbf{E_0}$, $\mathbf{H_0}$ be an electromagnetic wave with a harmonic time dependence of the form $e^{-i\omega t}$. We consider diffraction of the wave $\mathbf{E_0}$, $\mathbf{H_0}$ on the system Θ. The source of the incident field $\mathbf{E_0}$, $\mathbf{H_0}$ can be a current $j_{0,E}$ localized in such a way that $supp(j_{0,E}) \cap (\bar{Q} \bigcup \bar{\Omega}) = \varnothing$.

We look for the (complete) electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfying: the Maxwell equations

$$\begin{cases} \operatorname{rot} \mathbf{H} = -i\omega\hat{\varepsilon}\mathbf{E} + j_{0,E}, \\ \operatorname{rot} \mathbf{E} = i\omega\mu_e\mathbf{H}, \end{cases} \tag{1}$$

in $x \in \mathbb{R}^3 \setminus (\bar{Q} \bigcup \bar{\Omega})$, continuity conditions for the tangential components on the boundary

$$[\mathbf{E}_\tau]|_{\partial Q} = [\mathbf{H}_\tau]|_{\partial Q} = 0, \tag{2}$$

the boundary conditions in the internal points of Ω

$$\mathbf{E}_\tau|_{\partial\Omega} = 0, \tag{3}$$

the energy finiteness condition in any bounded domain

$$\mathbf{E}, \mathbf{H} \in L_{2,loc}(\mathbb{R}^3) \tag{4}$$

and the Sommerfeld radiation condition at infinity

$$\frac{\partial\left(\mathbf{E}_s, \mathbf{H}_s\right)}{\partial r} - ik_e\left(\mathbf{E}_s, \mathbf{H}_s\right) = o\left(\frac{1}{r}\right),\ r \to \infty \tag{5}$$

$\mathbf{E}_s = \mathbf{E} - \mathbf{E}_0$ and $\mathbf{H}_s = \mathbf{H} - \mathbf{H}_0$ is the scattered field, $r =\mid x \mid$, $x \in \mathbb{R}^3$.

The following uniqueness theorem is proved in [11].

Theorem 1. *The diffraction problem* (1) – (5) *has not more than one quasi-classical solution.*

2 System of Integro-differential Equations

The problem (1)–(5) is reduced to the system of integro-differential equations [11]:

$$\begin{aligned} &\hat{\xi}\mathbf{J} - \left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_Q G(x,y)\mathbf{J}(y)dy - \\ &\quad - \frac{1}{i\omega\varepsilon_e}\left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_\Omega G(x,y)\mathbf{u}(y)ds_y = \mathbf{E}_{0,Q}(x), \qquad x \in Q \\ &\Bigg(- \left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_Q G(x,y)\mathbf{J}(y)dy - \\ &\quad - \frac{1}{i\omega\varepsilon_e}\left({k_e}^2 + \operatorname{grad}\operatorname{div}_\tau\right)\int\limits_\Omega G(x,y)\mathbf{u}(y)ds_y\Bigg)_\tau = \mathbf{E}_{\mathbf{0},\tau}(x), \quad x \in \Omega, \end{aligned} \tag{6}$$

where $\hat{\xi} = \left(\frac{\hat{\varepsilon}(x)}{\varepsilon_e}^{-1} - \hat{I}\right)$, $\mathbf{J} = \left(\frac{\hat{\varepsilon}(x)}{\varepsilon_e} - \hat{I}\right)\mathbf{E}$ is the unknown polarization current vector in Q, $k_e = \omega\sqrt{\varepsilon_e, \mu_e}$ is the wave number of a free space, $\operatorname{Im}\varepsilon_e \geq 0$, $\operatorname{Im}\mu_e \geq 0$, $\operatorname{Im}k_e \geq 0$, $G(x,y) = \frac{1}{4\pi}\frac{e^{ik_e|x-y|}}{|x-y|}$ is the known Green function, $\mathbf{u}$ is an unknown surface current density on Ω, $\mathbf{E}_{0,\tau}$ is the tangential component of the incident field on the screen Ω.

Rewrite (6) in the operator form as follows:

$$\hat{L}(\mathbf{V}) = \mathbf{f}; \tag{7}$$

where

$$\hat{L} = \hat{L}_1 + \hat{L}_2 = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix},$$

the operators A, S, K_1 and K_2 are defined as follows:

$$A\mathbf{J} := \hat{\xi}\mathbf{J}(x) - \left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_Q G(x,y)\mathbf{J}(y)dy,$$

$$S\mathbf{u} := \left(-\frac{1}{i\omega\varepsilon_e}\left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_\Omega G(x,y)\mathbf{u}(y)ds_y\right)_\tau,$$

$$K_1\mathbf{u} := -\frac{1}{i\omega\varepsilon_e}\left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_\Omega G(x,y)\mathbf{u}(y)ds_y,$$

$$K_2\mathbf{J} := \left(-\left({k_e}^2 + \operatorname{grad}\operatorname{div}\right)\int\limits_Q G(x,y)\mathbf{J}(y)dy\right)_\tau,$$

$\mathbf{V} = (\mathbf{J}, \mathbf{u})$, the right-hand side is the vector $\mathbf{f} = (\mathbf{E}_{0,Q}, \mathbf{E}_{0,\tau})$, where $\mathbf{E}_{0,Q}$ is the restriction of the incident field.

Using non-coinciding integration points, we avoid singularity in the Green function.

3 Numerical Method

We suggest a numerical algorithm for solving equations (6). The equation (6) is discretized. Let the system Θ consist of a plane rectangular screen Ω, and a rectangular parallelepiped Q. We construct on the system Θ a generalized computational grid [6]. The computational grid is regular. Generalized computational grids allow us to introduce basis functions of different types [6].

We divide the screen Ω and the body Q into elementary cells that are called finite elements. For the screen Ω, the finite elements are rectangles

$$P_{k_1k_2} = \{x = (x_1, x_2),\ k_lh_l < x_l < (k_l+1)h_l,\ l = 1, 2\},$$

where $k = (k_1, k_2)$, $k_l = 0, \ldots, n-1$.

For the body Q the finite elements are rectangular parallelepipeds

$$\Pi^1_{j_1 j_2 j_3} = \{x : x_{1,j_1-1} < x_1 < x_{1,j_1+1},\ x_{2,j_2} < x_2 < x_{2,j_2+1},\ x_{3,j_3} < x_3 < x_{3,j_3+1}\},$$
$$x_{1,j_1} = a_1 + \frac{a_2 - a_1}{n} j_1,\ x_{2,j_2} = b_1 + \frac{b_2 - b_1}{n} j_2,\ x_{3,j_3} = c_1 + \frac{c_2 - c_1}{n} j_3,$$

where $j_1 = 0, \ldots, n-2$; $j_2, j_3 = 0, \ldots, n-1$. The number of supports on the screen Ω is $N_1 = 2n(n-1)$, the number of supports on the body Q is $N_2 = 3n^2(n-1)$. Thus, the number of supports for the system Θ is $N = N_1 + N_2$.

Let us consider the screen Ω. As the basis functions, we will use the "rooftop" functions, which are defined on a pair of adjacent rectangles of the grid with a shared edge. For each edge k there exists a support consisting of two rectangles with a shared edge k. The basis function $\varphi(x_1, x_2, x_3)$ corresponding to the edge k is determined as follows

$$\varphi_k(x_1, x_2, x_3) = \begin{cases} (x_1 - x_{1,k-1}, x_2 - x_{2,k-1}, x_3 - x_{3,k-1}) \frac{l_k}{S^+_k} \text{ in } P^+_k, \\ (x_{1,k+1} - x_1, x_{2,k+1} - x_2, x_{3,k+1} - x_3) \frac{l_k}{S^-_k} \text{ in } P^-_k, \end{cases}$$

where l_k is the length of the edge k, S^+_k and S^-_k are areas of P^+_k and P^-_k, respectively.

Let us consider the body Q. Supports of basis functions are pairs of adjacent elementary parallelepipeds belonging to the body and having a shared face. The parallelepipeds of a support are located along one of the coordinate axes.

We define basis functions on the body. Let $h^1 := |x_{1,j_1} - x_{1,j_1-1}|$. The function $\psi^1_{j_1,j_2,j_3}$ is defined as follows

$$\psi^1_{j_1,j_2,j_3}(x_1, x_2, x_3) = \begin{cases} 1 - \frac{1}{h^1}|x_1 - x_{1,j_1}| & \text{in } \Pi^1_{j_1,j_2,j_3} \\ 0, & \text{otherwise.} \end{cases}$$

The functions $\psi^2_{j_1,j_2,j_3}$ and $\psi^3_{j_1,j_2,j_3}$ are defined in the same way. The function $\psi^i_{j_1,j_2,j_3}$ is piecewise-linear in the Ox_i direction and piecewise-constant in two other directions.

Applying the Galerkin method [12] to the system (7), we obtain the equation

$$\mathbf{L}\mathbf{V} = \mathbf{f}; \tag{8}$$

where $\mathbf{L}$ is the generalized matrix of system of linear algebraic equations (SLAE), $\mathbf{V} = (\mathbf{J}, \mathbf{u})$ is the column of unknown coefficients for basis functions, $\mathbf{f}$ is a known.

The matrix of the SLAE has a block form

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} \\ \mathbf{L}^{21} & \mathbf{L}^{22} \end{pmatrix}.$$

The matrix block $\mathbf{L}^{11}$ corresponds to the solution of the diffraction problem only on the body. The matrix block $\mathbf{L}^{22}$ corresponds to the solution of the diffraction problem only on the screen.

The matrix elements, which correspond to the solution of the problem on the body Q , are obtained by calculating threefold and sixfold integrals over the body

$$\mathbf{L}_{pq}^{11} = \int\limits_{Q} \hat{\xi}(x)\boldsymbol{\psi}_q(x)\boldsymbol{\psi}_p(x)dx - \int\limits_{Q} \left(k_e^2 + \mathrm{grad}_x\, \mathrm{div}_x\right) \int\limits_{Q} G(x,y)\boldsymbol{\psi}_q(y)dy\boldsymbol{\psi}_p(x)dx.$$

The matrix elements, which correspond to the solution of the problem on the screen Ω , are obtained by calculating fourfold integrals over the screen

$$\mathbf{L}_{pq}^{22} = \frac{1}{\omega^2\varepsilon_e^2} \int\limits_{\Omega} \left(\left(k_e^2 + \mathrm{grad}_x\, \mathrm{div}_x\right) \int\limits_{\Omega} G(x,y)\boldsymbol{\varphi}_q(y)ds_y \right)_{\tau} \boldsymbol{\varphi}_p(x)ds_x.$$

The blocks $\mathbf{L}^{12}$ and $\mathbf{L}^{21}$ correspond to the interaction of fields on the body and screen; the matrix elements in these blocks are obtained by calculating fivefold integrals over the body and the screen

$$\mathbf{L}_{pq}^{12} = -\frac{1}{i\omega\varepsilon_e} \int\limits_{Q} \left(k_e^2 + \mathrm{grad}_x\, \mathrm{div}_{\tau,x}\right) \int\limits_{\Omega} G(x,y)\boldsymbol{\varphi}_q(y)ds_y\boldsymbol{\psi}_p(x)dx,$$

$$\mathbf{L}_{pq}^{21} = -\frac{1}{i\omega\varepsilon_e} \int\limits_{\Omega} \left(\left(k_e^2 + \mathrm{grad}_x\, \mathrm{div}_x\right) \int\limits_{Q} G(x,y)\boldsymbol{\psi}_q(y)dx \right)_{\tau} \boldsymbol{\varphi}_p(x)ds_y.$$

The generalized matrix is symmetrical. The block $\mathbf{L}^{21}$ coincides with the transposed block $\mathbf{L}^{12}$: $\mathbf{L}^{21} = \left(\mathbf{L}^{12}\right)^T$.

4 Numerical Results

An example of graphical results of calculations for a system of an inhomogeneous parallelepiped body and a rectangular screen (figure 2) is presented. The wave number k_e is 1. The size of the computational grid for each axis are 24 steps for the screen and 13 steps for the body.

The screen is located in the plane Ox_1x_2, $(x_3 = 0)$. The center of the body coincides with the center of the coordinate system. Thus

$$\Omega = \left\{ x \in \mathbb{R}^3 : x_1, x_2 \in \left(-\frac{\lambda}{2}, \frac{\lambda}{2} \right);\ x_3 = 0 \right\},$$

$$Q = \left\{ x \in \mathbb{R}^3 : x_k \in \left(-\frac{\lambda}{4}, \frac{\lambda}{4} \right) \right\}, \quad k = 1, 2, 3.$$

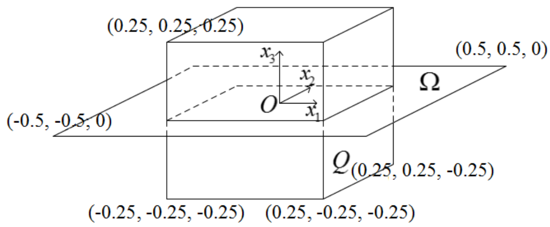

Fig. 2. The system Θ.

The incident field is a plane wave; its direction vector of which is collinear to the axis Ox_1. The body Q consists of two parts with different permittivities. The integration accuracy is 8 knots per support, λ is wavelength.

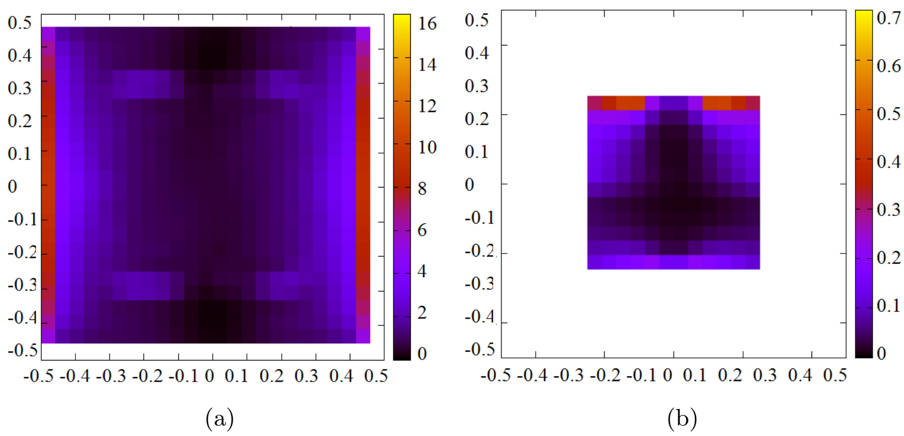

Fig. 3. Results of solving the diffraction problem on the system Θ on the different computation grids.

Figure 3(a) illustrates the distribution of surface current modules on the screen. Figure 3(b) illustrates the distribution of the field inside the body Q on the middle layer of the computational grid perpendicular to the axis Ox_1.

Figure 3(a) shows that the tangential component of the surface currents increases at the boundary of the intersection of the body and screen. Figure 3(b) shows the "jump" of the field inside the body at the interface of the two parts with different permittivities. Also figure 3b shows that the field inside the body increases at the boundary of the intersection of the body and screen.

We will use the subhierarchical method [6] to solve the diffraction problem one a system of inhomogeneous bodies and screens of complex shapes. Following the proposed method, at the first step the diffraction problem is solved most accurately on the canonical system of bodies and screens. The generalized computational grid is used. In the next step, using the obtained matrix SLAE, we select a new system of bodies and screens, or only bodies, or only screens that is contained in the canonical system. Further, without repeating calculations in the SLAE matrix, we find the solution of the system of integro-differential equations corresponding to the new problem.

Now let us consider the systems Θ_1 consisting of the screen Ω, and the system Θ_2 consisting of the body Q. Thus $\Theta = \Theta_1 \cup \Theta_2$.

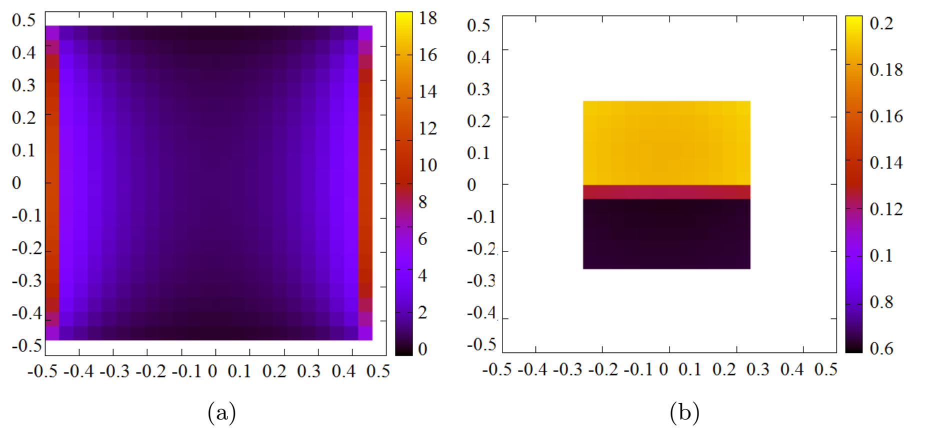

Fig. 4. Results of solving the diffraction problem on Θ_1 and Θ_2 on the different computation grids.

Figure 4(a) illustrates the distribution of surface current modules on the screen, figure 4(b) illustrates the distribution of the field inside the body Q on the middle layer of the computed grid perpendicular to the axis Ox_1.

The amount of memory allocated to the generalized matrix is about 2 GB. The results presented in the articles [6–8] for a screen and in the article [13] for a body are in good agreement the results presented in figure 4a and 4b. We observe a qualitative coincidence of the results on the graphs presented in figures 3(a)–3(b) and 4(a)–4(b), with increasing grid steps on the system. This corresponds to the internal convergence of the solutions.

5 Parallel Algorithm

Due to large computations we use a parallel algorithm. About 80% of time is used to the calculations of the matrix entries, which are calculated independently. The matrices have about 10^4 entries that are threefold and sixfold volume integrals, fourfold surface integrals, fourfold volume and surface integrals. The main computational complexity is to calculate the matrix block $\mathbf{L}^{11}$.

Parallel algorithm:

1. We convert the matrix into a 1-dimensional array $\{A_I\}$ of size N^2;
2. It is necessary to uniformly distribute coefficients C of the array $\{A_I\}$ which are computed at each process [14]:

$$C = \begin{cases} \left[\frac{N}{p}\right] + 1, & p < \left\{\frac{N}{p}\right\}, \\ \left[\frac{N}{p}\right], & p \geqslant \left\{\frac{N}{p}\right\}, \end{cases}$$

where p is the process' number, $\left[\frac{N}{p}\right]$ is an integer part of the ratio, $\left\{\frac{N}{p}\right\}$ is a residue of integer division.

If the computational grid becomes more dense, then the data level, which is sent between cores, increases proportionally. Neglecting this fact, one can get almost zero speed gain. For this reason, it is necessary to pay attention to the time required for data transfer. If a processor does not respond during a long period, the data must be redistribute between other cores. Size S of the matrix $\mathbf{L}$ is calculated as follows

$$\begin{aligned} S_{11} &:= \mathrm{size}(\mathbf{L}^{11}) = \left(3n^2(n-1)\right)^2, \\ S_{12} &:= \mathrm{size}(\mathbf{L}^{12}) = \left(3n^2(n-1)\right)\left(2n(n-1)\right), \\ S_{21} &:= \mathrm{size}(\mathbf{L}^{21}) = \left(3n^2(n-1)\right)\left(2n(n-1)\right), \\ S_{22} &:= \mathrm{size}(\mathbf{L}^{22}) = \left(2n(n-1)\right)^2, \end{aligned}$$

$$S := S_{11} + S_{12} + S_{21} + S_{22}.$$

It is seen that memory capacity needed for solution to such a problem increases drastically. For example, with $n = 10$ the matrix requires 124 Mb memory, with $n = 20$ the matrix requires 8.3 Gb memory, with $n = 30$ the matrix requires 95 Gb memory.

Below we show results of speed gain for different number of cores:

1. with $p = 8$ the computation speed increases approximately 8 times;
2. with $p = 16$ the computational speed increases approximately 15 times;
3. with $p = 32$ the computational speed increases approximately 26 times;
4. with $p = 64$ the computational speed increases approximately 40 times.

The parallel algorithm is quite simple in use and allows reducing calculating time up to 40 times. We used a 64-core processor to solve the problem.

Conclusion

The method of solving the diffraction problem on a system of intersecting dielectric bodies and screens of complex shape is suggested. To solve the problem, the computational algorithm is parallelized, the supercomputer complex of the Penza State University was used for calculations. The methods proposed in this paper can be used in practice to study the behavior of the field reflected from intersecting (dielectric) bodies and (infinitely thin perfectly conducting) screens. Numerical results are presented and visualized. In the neighborhood of the screen edge and the body surface the behavior of the reflected field is in qualitatively agreement with the theory [5, 13].

Acknowledgments

This study is supported by the Ministry of Education and Science of the Russian Federation [project 1.894.2017/4.6] and by the Russian Foundation for Basic Research [project 18-31-00108]. The research is carried out using the equipment of the shared research facilities of HPC computing resources at Lomonosov Moscow State University.

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